ELSEVIER

Journal of Geometry and Physics 37 (2001) 262–271

JOURNAL OF GEOMETRY AND PHYSICS

# Reduction for locally conformal symplectic manifolds

Stefan Haller [a,1], Tomasz Rybicki [b,*]

[a] *Institute of Mathematics, University of Vienna, Strudlhofg. 4, A-1090 Vienna, Austria*
[b] *Department of Applied Mathematics at AGH, AL. Mickiewicza 30, 30-059 Cracow, Poland*

Received 16 May 2000

**Abstract**

It is shown how one can do symplectic reduction for locally conformal symplectic manifolds, especially with an action of a Lie group. This generalizes well-known procedures for symplectic manifolds to the slightly larger class of locally conformal symplectic manifolds. The whole setting is very conformally invariant. 

*MSC:* 53D20

*Subj. Class.:* Dynamical systems; Differential geometry

*Keywords:* Locally conformal symplectic manifolds; Symplectic reduction

## 1. Introduction

The geometry of locally conformal symplectic (l.c.s. for short) structures generalizes the symplectic geometry. Such structures occur naturally in the theory of Jacobi manifolds, as any even-dimensional leaf of the distinguished foliation of a Jacobi manifold is l.c.s., cf. [2]. In particular, any transitive even-dimensional Jacobi manifold is actually an l.c.s. manifold. The l.c.s. manifolds may be regarded as natural phase spaces in the Hamiltonian mechanics [8]. For instance, in [9], the symmetry of the Lyapunov spectrum in locally conformal Hamiltonian systems is studied. The authors of [9] present several examples and show that Gaussian isokinetic dynamics, Nosé–Hoovers dynamics and other systems can be viewed as locally conformal Hamiltonian systems.

* Corresponding author.
*E-mail addresses:* stefan@mat.univie.ac.at (S. Haller), tomasz@uci.agh.edu.pl (T. Rybicki).
[1] Present address: Department of Mathematics, Ohio State University, 231 West 18th Avenue, Columbus, OH 43210, USA.

0393-0440/01/$ – see front matter 
PII: S0393-0440(00)00050-4

The aim of this note is to show that, in regular cases, the symplectic reduction is possible for l.c.s. manifolds. The most abstract reduction principle states that every coisotropic submanifold of a symplectic manifold is foliated by isotropic leaves and if the leaf space is a manifold it carries again a symplectic structure. In the paper of Marsden and Weinstein [7], it has been formalized the fact that if an $n$-dimensional symmetry group acts on a Hamiltonian system, then the number of degrees of freedom can be reduced by $n$, and the dimension of the phase space is reduced by $2n$ (the position and the momentum coordinates). We show that this is still true for l.c.s. manifolds. Moreover, the reduction process preserves the conformal equivalence class of an l.c.s. structure. An open problem is how to treat singular reductions (cf. [1]) in the l.c.s. case.

## 2. Locally conformal symplectic manifolds

For a closed 1-form $\omega \in \Omega^1(M)$ and a vector field $X \in \mathfrak{X}(M)$ we define $\mathrm{d}^\omega : \Omega^*(M) \to \Omega^{*+1}(M)$ by $\mathrm{d}^\omega\alpha := \mathrm{d}\alpha + \omega\wedge\alpha$, and $L_X^\omega : \Omega^*(M) \to \Omega^*(M)$ by $L_X^\omega\alpha := L_X\alpha + \omega(X)\alpha$. The well-known formulas $\mathrm{d}^\omega \mathrm{d}^\omega = 0$, $L_X^\omega L_Y^\omega - L_Y^\omega L_X^\omega = L_{[X,Y]}^\omega$, $L_X^\omega \mathrm{d}^\omega - \mathrm{d}^\omega L_X^\omega = 0$, $L_X^\omega i_Y - i_Y L_X^\omega = i_{[X,Y]}$ and $\mathrm{d}^\omega i_X + i_X \mathrm{d}^\omega = L_X^\omega$ remain true. Moreover, one has the derivation like formulas

$$\mathrm{d}^{\omega_1+\omega_2}(\alpha\wedge\beta) = \mathrm{d}^{\omega_1}\alpha\wedge\beta + (-1)^{|\alpha|}\alpha\wedge \mathrm{d}^{\omega_2}\beta,$$
$$L_X^{\omega_1+\omega_2}(\alpha\wedge\beta) = L_X^{\omega_1}\alpha\wedge\beta + \alpha\wedge L_X^{\omega_2}\beta.$$

An l.c.s. manifold is a triple $(M, \Omega, \omega)$, where $\Omega \in \Omega^2(M)$ is non-degenerate, $\omega \in \Omega^1(M)$ is closed and $\mathrm{d}^\omega\Omega = 0$. If $\dim(M) > 2$, then $\omega$ is uniquely determined by $\Omega$. Two l.c.s. structures $(\Omega, \omega)$ and $(\Omega', \omega')$ are called conformally equivalent (we will write $(\Omega, \omega) \sim (\Omega', \omega')$) if there exists a positive function $a$ such that $\Omega' = (1/a)\Omega$ and $\omega' = \omega + (\mathrm{d}a/a) = \omega + \mathrm{d}\ln a$. Notice that, if $\dim(M) > 2$, the second equation is a consequence of the first.

Suppose, we have a manifold $M$, an open covering $\mathcal{U}$ of $M$ and on every $U \in \mathcal{U}$ an l.c.s. structure $(\Omega_U, \omega_U)$, such that $(\Omega_U, \omega_U)|_{U\cap V} \sim (\Omega_V, \omega_V)|_{U\cap V}$ for all $U, V \in \mathcal{U}$. Then there exists, up to conformal equivalence, a unique l.c.s. structure $(\Omega, \omega)$ on $M$ with $(\Omega, \omega)|_U \sim (\Omega_U, \omega_U)$ for all $U \in \mathcal{U}$. It can be constructed as follows. For every pair $U$, $V \in \mathcal{U}$ there exists a function $a_{UV} : U \cap V \to \mathbb{R}^+$, such that $\Omega_U|_{U\cap V} = (1/a_{UV})\Omega_V|_{U\cap V}$ and $\omega_U|_{U\cap V} = \omega_V|_{U\cap V} + (\mathrm{d}a_{UV}/a_{UV})$. Obviously, we have $a_{UV}a_{VW} = a_{UW}$ on $U\cap V\cap W$. In other words, $\{a_{UV}\}$ is a 1-cocycle in the Čech-cohomology of the sheaf $C^\infty(\cdot, \mathbb{R}^+)$. Since this sheaf is fine, and hence acyclic, every cocycle is a coboundary, i.e. there exist functions $b_U : U \to \mathbb{R}^+$ with $a_{UV} = b_V/b_U$ on $U \cap V$. Now one defines $\Omega|_U := (1/b_U)\Omega_U$ and $\omega|_U := \omega_U + (\mathrm{d}b_U/b_U)$ [5].

By $\mathrm{Diff}^\infty(M, \Omega, \omega)$ we denote the group of all diffeomorphisms $M$, which preserve $(\Omega, \omega)$ up to conformal equivalence. The corresponding Lie algebra of vector fields is

$$\mathfrak{X}(M, \Omega, \omega) = \{X \in \mathfrak{X}(M) : L_X^\omega\Omega = c_X\Omega \text{ for some locally constant } c_X\},$$

cf. [4].

$C^\infty(M,\mathbb{R})$ becomes a Lie algebra with the following bracket

$$\{f_1, f_2\} := \Omega(\sharp\, \mathrm{d}^\omega f_1, \sharp\, \mathrm{d}^\omega f_2) = L^\omega_{\sharp\, \mathrm{d}^\omega f_2} f_1 = -L^\omega_{\sharp\, \mathrm{d}^\omega f_1} f_2.$$

Here $\sharp$ denotes the inverse of $\flat$, where $\flat : TM \cong T^*M$ given by $X \mapsto i_X\Omega$ is a canonical vector bundle isomorphism since $\Omega$ is non-degenerate. We have $\sharp\, \mathrm{d}^\omega\{f_1, f_2\} = -[\sharp\, \mathrm{d}^\omega f_1, \sharp\, \mathrm{d}^\omega f_2]$. So

$$\mathrm{Ham}(M,\Omega,\omega) := \{\sharp\, \mathrm{d}^\omega f : f \in C^\infty(M,\mathbb{R})\}$$

is a subalgebra of $\mathfrak{X}(M,\Omega,\omega)$, and

$$H^0_{\mathrm{d}^\omega}(M) \hookrightarrow C^\infty(M,\mathbb{R}) \xrightarrow{\sharp\, \mathrm{d}^\omega} \mathrm{Ham}(M,\Omega,\omega)$$

is, up to a sign, a central extension of Lie algebras. Here $H^*_{\mathrm{d}^\omega}(M)$ denotes the cohomology with respect to the differential $\mathrm{d}^\omega$, especially

$$H^0_{\mathrm{d}^\omega}(M) = \{f \in C^\infty(M,\mathbb{R}) : \mathrm{d}^\omega f = 0\}.$$

Note that if $(\Omega', \omega') \sim (\Omega, \omega)$, i.e. $\Omega' = (1/a)\Omega$ and $\omega' = \omega + (\mathrm{d}a/a)$, then $\{f, g\}' = (1/a)\{af, ag\}$. Moreover, one has $\{fg, h\} = f\{g, h\} + \{f, h\}g - fgL_{\sharp\omega}h$.

## 3. Reduction for l.c.s. structures

We begin with the following general definition of reduction essentially due to Marsden and Ratiu [6].

**Definition.** Let $(M, \Omega, \omega)$ be an l.c.s. manifold, and let $\{\cdot,\cdot\}_M$ be the corresponding bracket on $C^\infty(M,\mathbb{R})$. A pair $(L, E)$, where $i : L \subset M$ is a submanifold and $E$ is a vector subbundle of $TM|_L$, is said to be a *reductive structure* if the following is fulfilled:

1. $E \cap TL$ is tangent to a foliation of $L$ whose leaves are the fibers of a submersion $\pi : L \to P_L$;
2. $i^*\omega(X) = 0$ for any $X$ tangent to the fibers of $\pi$;
3. for any $u, v \in C^\infty(M,\mathbb{R})$ such that $\mathrm{d}u, \mathrm{d}v$ vanish on $E$ $\mathrm{d}^\omega\{u, v\}_M$ vanishes on $E$ as well.

Furthermore, $(M, L, E)$ is called a *reducible triple* if there is an l.c.s. structure $(\bar\Omega, \bar\omega)$ on $P_L$ with the Poisson bracket $\{\cdot,\cdot\}_{P_L}$ such that for any local $C^\infty$ functions $f, g$ on $P_L$, and any local extensions $u, v$ of $f \circ \pi$, $g \circ \pi$, respectively, such that both $\mathrm{d}^\omega u$ and $\mathrm{d}^\omega v$ vanish on $E$, one has the relation

$$\{u, v\}_M \circ i = \{f, g\}_{P_L} \circ \pi.$$

$(P_L, \bar\Omega, \bar\omega)$ is then a *reduced l.c.s. manifold*.

A standard reductive structure arises in the following situation. Let $(M, \Omega, \omega)$ be an l.c.s. manifold such that for a submanifold $i : L \subset M$ $\mathrm{rank}(i^*\Omega)$ is constant. Then, it is

apparent that for the $\Omega$-orthogonal space $TL^\perp$ of $TL$, $TL^\perp \cap TL$ is integrable. Hence, if the resulting foliation is given by a submersion $\pi : L \to P_L$ and $i^*\omega$ vanishes on $TL^\perp \cap TL$ then $(L, TL^\perp \cap TL)$ is a reductive structure. By Proposition 1 below we see that $(M, L, TL^\perp \cap TL)$ is a reducible triple as well since $\sharp\,\mathrm{Ann}(TL^\perp) = (TL^\perp)^\perp = TL$.

As in the symplectic and Poisson case [6], we have the following proposition.

**Proposition 1.** *Let $(M, \Omega, \omega)$ be an l.c.s. manifold and suppose $(L, E)$ is a reductive structure. Then $(M, L, E)$ is a reducible triple if and only if*

$$\sharp(\mathrm{Ann}\, E) \subset TL + E. \tag{1}$$

*Moreover, if $(M, \Omega, \omega) \sim (M, \Omega', \omega')$ then $(P_L, \bar{\Omega}, \bar{\omega}) \sim (P_L, \bar{\Omega}', \bar{\omega}')$, i.e. the conformal equivalence relation is preserved.*

**Proof.** Assume that $(P_L, \bar{\Omega}, \bar{\omega})$ is a reduced l.c.s. of $(M, \Omega, \omega)$. Let $x \in L$, $\alpha \in \mathrm{Ann}\, E_x$, and $\beta \in \mathrm{Ann}(E_x + T_xL) = (\mathrm{Ann}\, E_x) \cap (\mathrm{Ann}\, T_xL)$. By (1) there is $u \in C^\infty(M, \mathbb{R})$ such $\mathrm{d}_x u = \alpha$ and $\mathrm{d}u|_E = 0$. Now due to (2) $\mathrm{d}^\omega_x u = \alpha$ and $\mathrm{d}^\omega u$ vanishes on $E$. Likewise, we choose $v \in C^\infty(M, \mathbb{R})$ vanishing on $L$ with $\mathrm{d}_x v = \beta$ and $\mathrm{d}v|_E = 0$. Hence in view of (2) $\mathrm{d}^\omega_x v = \beta$ and $\mathrm{d}^\omega v$ vanishes on $E$. Consequently, by the definition of reducible triple, we have

$$\begin{aligned}\beta(\sharp\alpha) &= (\mathrm{d}^\omega_x v)(\sharp\, \mathrm{d}^\omega_x u)\\ &= (\mathrm{d}_x v)(\sharp\, \mathrm{d}^\omega_x u) = (L_{\sharp\, \mathrm{d}^\omega_x u} v)(x) = \{v, u\}_M(x) = \{0, f\}(\pi(x))_{P_L} = 0,\end{aligned}$$

where $f \in C^\infty(P_L, \mathbb{R})$ such that $u$ extends $f \circ \pi$. This gives Eq. (1).

Conversely, suppose Eq. (1) is fulfilled and let $f, g \in C^\infty(P_L, \mathbb{R})$. For fixed $x \in L$ choose extensions $u, v \in C^\infty(M, \mathbb{R})$ of $f \circ \pi$, $g \circ \pi$, respectively, such that $u(x) = v(x) = 0$ and $\mathrm{d}u|_E = \mathrm{d}v|_E = 0$. This yields $\mathrm{d}\{u, v\}_M|_E = \mathrm{d}^\omega\{u, v\}_M|_E = 0$ and, consequently, $\{u, v\}_M|_L$ is constant along fibers. We let $\{f, g\}_{P_L}(\pi(x)) := \{u, v\}_M(x)$ which will ensure that $(M, L, E)$ is a reducible triple. The only thing to check is the independence of the choice of $u$ and $v$. If $\tilde{u}$ also extends $f \circ \pi$ with $\tilde{u}(x) = 0$ and $\mathrm{d}^\omega \tilde{u}|_E = 0$, we get $(u - \tilde{u})|_L = 0$. Therefore, $\mathrm{d}u = \mathrm{d}\tilde{u}$ on $E + TL$ which, in view of Eq. (1) implies

$$\{u, v\}_M(x) = (\mathrm{d}^\omega_x u)(\sharp\, \mathrm{d}^\omega_x v) = (\mathrm{d}^\omega_x \tilde{u})(\sharp\, \mathrm{d}^\omega_x v) = \{\tilde{u}, v\}_M(x).$$

By the antisymmetry, the definition is also independent of the choice of $v$. The second assertion is obvious. □

## 4. The moment mapping

Let $G$ be a finite dimensional Lie group acting from the left on $M$. We will write $l_g : M \to M$ for the action of the element $g \in G$. For the fundamental vector fields $\zeta_X$ we have $\zeta_{[X,Y]} = -[\zeta_X, \zeta_Y]$ and $\zeta_{\mathrm{Ad}_{g^{-1}} X} = l_g^* \zeta_X$. In the sequel we will always assume that $G$ acts symplectically, i.e. $l_g \in \mathrm{Diff}^\infty(M, \Omega, \omega)$ for all $g \in G$. If $G$ is connected this

is equivalent to $\zeta_X \in \mathfrak{X}(M, \Omega, \omega)$ for all $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of $G$. Then for every $g \in G$ there exists a positive function $a_g$ on $M$, such that $l_g^*\Omega = (1/a_g)\Omega$ and $l_g^*\omega = \omega + (\mathrm{d}a_g/a_g)$. Obviously, we have $a_{hg} = a_g(l_g^* a_h)$. So we have a right $G$-action on $C^\infty(M, \mathbb{R})$ given by $g \cdot f := a_g(l_g^* f)$. If one differentiates $l_g^*\Omega = (1/a_g)\Omega$ with respect to $X \in \mathfrak{g}$, one obtains $(\partial/\partial t)|_0 a_{\exp(tX)} = \omega(\zeta_X) - c_X$. So the corresponding $\mathfrak{g}$-action on $C^\infty(M, \mathbb{R})$ is $X \cdot f = L^\omega_{\zeta_X} f - c_X f$. Moreover, $\sharp\,\mathrm{d}^\omega : C^\infty(M, \mathbb{R}) \to \mathrm{Ham}(M, \Omega, \omega)$ is equivariant, i.e., $\sharp\,\mathrm{d}^\omega(a_g l_g^* f) = l_g^* \sharp\,\mathrm{d}^\omega f$ and $\sharp\,\mathrm{d}^\omega(L^\omega_{\zeta_X} f - c_X f) = [\zeta_X, \sharp\,\mathrm{d}^\omega f]$.

**Proposition 2.** *Let $G$ be a connected Lie group, and suppose the $\mathfrak{g}$-action is Hamiltonian, i.e. $\zeta_X \in \mathrm{Ham}(M, \Omega, \omega)$ for all $X \in \mathfrak{g}$. Let $\hat{\psi} : \mathfrak{g} \to C^\infty(M, \mathbb{R})$ be a linear lift of $\zeta$, i.e. $\sharp\,\mathrm{d}^\omega \hat{\psi}(X) = \zeta_X$ for all $X \in \mathfrak{g}$. Moreover, let $\psi : M \to \mathfrak{g}^*$ be defined by $\langle \psi(x), X\rangle = \hat{\psi}(X)(x)$, where $X \in \mathfrak{g}$ and $x \in M$. Then the following conditions*

1. *$\hat{\psi}$ is a homomorphism of Lie algebras,*
2. *$\hat{\psi}$ is equivariant, i.e. $\hat{\psi}(\mathrm{Ad}_{g^{-1}} X) = a_g l_g^* \hat{\psi}(X)$, for all $g \in G$, $X \in \mathfrak{g}$,*
3. *$\hat{\psi}$ is infinitesimal invariant, i.e. $\hat{\psi}(-\mathrm{ad}_Y X) = L^\omega_{\zeta_Y} \hat{\psi}(X)$ for all $X, Y \in \mathfrak{g}$,*
4. *$\mathrm{Ad}^*_{g^{-1}} \psi = a_g l_g^* \psi$ for all $g \in G$, and*
5. *$-\mathrm{ad}^*_Y \psi = T\psi \cdot \zeta_Y + \omega(\zeta_Y)\psi$ for all $Y \in \mathfrak{g}$*

*are equivalent.*

**Proof.** We have $\hat{\psi}(-\mathrm{ad}_Y X) = \hat{\psi}([X, Y])$ and

$$L^\omega_{\zeta_Y} \hat{\psi}(X) = L^\omega_{\sharp\,\mathrm{d}^\omega \hat{\psi}(Y)} \hat{\psi}(X) = \{\hat{\psi}(X), \hat{\psi}(Y)\}.$$

So (1) is equivalent to (3), and since $G$ is connected (3) is equivalent to (2). Notice that since the action is Hamiltonian we have $c_X = 0$ and so (3) is really the infinitesimal version of (2). Moreover, (4) is obviously equivalent to (5). Next we have $\hat{\psi}(-\mathrm{ad}_Y X)(x) = \langle -\mathrm{ad}^*_Y \psi(x), X\rangle$, and

$$\begin{aligned} L^\omega_{\zeta_Y} \hat{\psi}(X)(x) &= L_{\zeta_Y} \hat{\psi}(X)(x) + \omega(\zeta_Y(x))\hat{\psi}(X)(x) = L_{\zeta_Y} \langle \psi(x), X\rangle \\ &\quad + \omega(\zeta_Y(x))\langle \psi(x), X\rangle = \langle T_x\psi \cdot \zeta_Y(x), X\rangle + \langle \omega(\zeta_Y(x))\psi(x), X\rangle, \end{aligned}$$

and thus (5) is equivalent to (3). □

A mapping $\psi$ as in Proposition 2 is called a moment mapping, and it is called equivariant if the five equivalent conditions are satisfied.

**Proposition 3.** *Let $\varphi : \mathfrak{g} \to \mathrm{Ham}(M, \Omega, \omega)$ be an anti homomorphism of Lie algebras. If $H^2(\mathfrak{g}; H^0_{\mathrm{d}^\omega}(M)) = 0$, where $H^0_{\mathrm{d}^\omega}(M)$ is considered as trivial $\mathfrak{g}$-module, then there exists a lift $\tilde{\varphi} : \mathfrak{g} \to C^\infty(M, \mathbb{R})$, i.e. $\tilde{\varphi}$ is a homomorphism of Lie algebras and $\sharp\,\mathrm{d}^\omega \circ \tilde{\varphi} = \varphi$. Moreover, if there exists a lift, then the set of all lifts is parameterized by $H^1(\mathfrak{g}; H^0_{\mathrm{d}^\omega}(M))$.*

**Proof.** Let $\bar{\varphi} : \mathfrak{g} \to C^\infty(M, \mathbb{R})$ be a linear lift of $\varphi$ and set

$$c(X, Y) := \bar{\varphi}([X, Y]) - \{\bar{\varphi}(X), \bar{\varphi}(Y)\} \in H^0_{\mathrm{d}^\omega}(M).$$

Then one easily shows $\delta c = 0$. Since $H^2(\mathfrak{g}; H^0_{\mathrm{d}^\omega}(M)) = 0$ there exists $b$ with $\delta b = c$, i.e. $c(X, Y) = (\delta b)(X, Y) = -b([X, Y])$. Now we define $\tilde{\varphi} : \mathfrak{g} \to C^\infty(M, \mathbb{R})$ by $\tilde{\varphi}(X) := \bar{\varphi}(X) + b(X)$. Then one immediately sees that $\tilde{\varphi}$ is a homomorphism of Lie algebras and a lift of $\varphi$.

To see the second assertion, let $\tilde{\varphi}$ denote a homomorphism of Lie algebras lifting $\varphi$. All other lifts are of the form $\tilde{\varphi} + b$ for some $b : \mathfrak{g} \to C^\infty(M, \mathbb{R})$. Since we have

$$\{\tilde{\varphi}(X) + b(X), \tilde{\varphi}(Y) + b(Y)\} = \{\tilde{\varphi}(X), \tilde{\varphi}(Y)\} = \tilde{\varphi}([X, Y]),$$

we see that $\tilde{\varphi} + b$ is a homomorphism of Lie algebras iff $b([X, Y]) = 0$ for all $X, Y \in \mathfrak{g}$, or equivalently $\delta b = 0$, i.e. $b \in H^1(\mathfrak{g}; H^0_{\mathrm{d}^\omega}(M))$. □

If $\mathfrak{g}$ is semi-simple, then there always exists an equivariant moment mapping, since $H^2(\mathfrak{g}; H^0_{\mathrm{d}^\omega}(M)) = 0$, by the second Whitehead lemma. Moreover, it is unique since $H^1(\mathfrak{g}; H^0_{\mathrm{d}^\omega}(M)) = 0$, by the first Whitehead lemma.

If $(M, \Omega, \omega)$ is an l.c.s. manifold which is not conformally equivalent to a symplectic manifold, i.e. $\omega$ is not exact, then there always exists a unique equivariant moment mapping. This is because $H^0_{\mathrm{d}^\omega}(M) = 0$ for non-exact $\omega$.

Suppose the l.c.s. structure is exact, i.e. $\Omega = \mathrm{d}^\omega\theta$, and the action preserves $\theta$, i.e. $L^\omega_{\zeta_X}\theta = c_X\theta$ for locally constant functions $c_X$. Notice that this immediately implies $L^\omega_{\zeta_X}\Omega = c_X\Omega$. Moreover, $c_X = 0$ iff the action is Hamiltonian, and in this situation $\hat{\psi}(X) = -i_{\zeta_X}\theta$ is an equivariant moment mapping.

## 5. Lie group actions and reduction

Let $G$ be a finite dimensional Lie group acting symplectically on an l.c.s. manifold $(M, \Omega, \omega)$. Assume that we have an equivariant moment mapping $\psi$. The vector fields $Z \in \mathfrak{X}(M)$ which are $\Omega$-orthogonal to the orbits of $G$ span an involutive distribution on $M$. Indeed, since we have $\Omega(\zeta_X, Z) = i_Z\,\mathrm{d}^\omega\hat{\psi}(X) = L^\omega_Z\hat{\psi}(X)$, this follows from $L^\omega_{[Z_1,Z_2]} = L^\omega_{Z_1}L^\omega_{Z_2} - L^\omega_{Z_2}L^\omega_{Z_1}$. Suppose $L$ is a maximal connected submanifold of $M$ which is tangent to this distribution, more precisely $T_xL = \zeta_{\mathfrak{g}}(x)^\perp$ for all $x \in L$. Since we have $0 = (L^\omega_Z\hat{\psi}(X))(x) = \langle T_x\psi \cdot Z_x + \omega(Z_x)\psi(x), X\rangle$, we see that $\psi(L) \subseteq \mathbb{R}^+ \cdot \psi(x_0)$ for any $x_0 \in L$. However $\psi$ need not be constant along $L$.

Let $\mathfrak{g}_L := \{Y \in \mathfrak{g} : \mathrm{ad}^*_Y\psi(x_0) = 0\}$, which does not depend on $x_0 \in L$. By the equivariance of $\psi$, these are precisely those $Y$ for which $\zeta_Y$ is tangential to $L$. Since $L$ is maximal, the connected subgroup of $G$ corresponding to $\mathfrak{g}_L$ leaves $L$ invariant.

**Theorem 1.** *Let $G$ be a finite dimensional Lie group acting symplectically on an l.c.s. manifold $(M, \Omega, \omega)$ and assume that the action admits an equivariant moment mapping $\psi$. Suppose $L$ is a maximal connected submanifold of $M$ with $T_xL = \zeta_{\mathfrak{g}}(x)^\perp$. Let $G_L$ be a subgroup of $G$ which preserves $L$ and has $\mathfrak{g}_L$ as Lie algebra, and assume that $G_L$ acts freely and properly on $L$. Then $P_L := L/G_L$ admits a unique (up to conformal equivalence) l.c.s. structure $(\bar{\Omega}, \bar{\omega})$, such that $(L, i^*\Omega, i^*\omega) \sim (L, \pi^*\bar{\Omega}, \pi^*\bar{\omega})$, where $i : L \hookrightarrow M$ denotes the inclusion and $\pi : L \to P_L$ denotes the projection.*

**Proof.** Since the action is free and proper $\pi : L \to P_L$ is a principle $G_L$-bundle. Let $\mathcal{U}$ be an open covering of $P_L$ and for $U \in \mathcal{U}$, let $s_U : U \to L$ be local sections of $\pi$. Define $\bar{\Omega}_U := s_U^* i^* \Omega$ and $\bar{\omega}_U := s_U^* i^* \omega$. We claim that $(U, \bar{\Omega}_U, \bar{\omega}_U)$ is l.c.s. for every $U \in \mathcal{U}$. Obviously $\mathrm{d}\bar{\omega}_U = 0$ and $\mathrm{d}^{\bar{\omega}_U} \bar{\Omega}_U = 0$. To see that $\bar{\Omega}_U$ is non-degenerate, let $Y_z \in T_z P_L$ and suppose $0 = \bar{\Omega}_U(Y_z, Z_z) = i^*\Omega(T_z s_U \cdot Y_z, T_z s_U \cdot Z_z)$ for all $Z_z \in T_z P_L$. Since $i_{\zeta_X} i^* \Omega = 0$ for all $X \in \mathfrak{g}_L$, we conclude that $\Omega(T_z s_U \cdot Y_z, T_{s(z)} L) = 0$. In other words, $T_z s_U \cdot Y_z \in (T_{s_U(z)} L)^\perp = \zeta_{\mathfrak{g}}(s_U(z))$ and thus $T_z s_U \cdot Y_z = 0$, i.e. $Y_z = 0$.

Next we will show that $(i^*\Omega, i^*\omega)|_{\pi^{-1}(U)} \sim \pi^*(\bar{\Omega}_U, \bar{\omega}_U)$. So define $f : \pi^{-1}(U) \to G$ by $l_{f(x)}(x) := s_U(\pi(x))$, and $a \in C^\infty(\pi^{-1}(U), \mathbb{R}^+)$ by $a(x) := a_{f(x)}(x)$. We claim that $\pi^*\bar{\Omega}_U = (1/a) i^*\Omega|_{\pi^{-1}(U)}$ and $\pi^*\bar{\omega}_U = i^*\omega|_{\pi^{-1}(U)} + (\mathrm{d}a/a)$. Indeed, we have $T_x(s_U\pi) \cdot Z_x = T_x l_{f(x)} \cdot Z_x + V$ for a vertical $V \in T_{s_U(\pi(x))} L$. Since $i^*\Omega$ vanishes on vertical vectors, we get

$$
\begin{aligned}
(\pi^*\bar{\Omega}_U)(Z_x, Z'_x) &= (s_U\pi)^* i^*\Omega(Z_x, Z'_x) = i^*\Omega(T_x(s_U\pi) \cdot Z_x, T_x(s_U\pi) \cdot Z'_x) \\
&= i^*\Omega(T_x l_{f(x)} \cdot Z_x, T_x l_{f(x)} \cdot Z'_x) = i^* l_{f(x)}^* \Omega(Z_x, Z'_x) = \frac{1}{a(x)} i^*\Omega(Z_x, Z'_x),
\end{aligned}
$$

i.e. $\pi^*\bar{\Omega}_U = (1/a) i^*\Omega|_{\pi^{-1}(U)}$. Moreover, we have $f(gx) = f(x)g^{-1}$ for $g \in G_L$, and therefore $a(gx) = (1/a_g(x))a(x)$, where we used $a_{hg}(x) = a_g(x) a_h(gx)$. From this, we obtain

$$
i_{\zeta_Y(x)}\, \mathrm{d}a = \left.\frac{\partial}{\partial t}\right|_0 \left(\frac{1}{a_{\exp(tY)}(x)} a(x)\right) = -i^*\omega(\zeta_Y(x)) a(x),
$$

i.e. $(\pi^*\bar{\omega}_U - i^*\omega - (\mathrm{d}a/a))(\zeta_Y) = 0$ for all $Y \in \mathfrak{g}_L$. Remains to check this equation on 'horizontal' vector fields. Since $f(s_U(z)) = e$ we have $s_U^* a = 1$ and thus

$$
(l_g s_U)^* \left(i^*\omega + \frac{\mathrm{d}a}{a}\right) = s_U^* \left(i^*\omega + \frac{\mathrm{d}a_g}{a_g} + \frac{\mathrm{d}l_g^* a}{l_g^* a}\right) = s_U^* \left(i^*\omega + \frac{\mathrm{d}a}{a}\right) = \bar{\omega}_U.
$$

Since $(l_g s_U)^*(\pi^*\bar{\omega}_U) = \bar{\omega}_U$, we obtain $\pi^*\bar{\omega}_U = i^*\omega|_{\pi^{-1}(U)} + (\mathrm{d}a/a)$. If $V \in \mathcal{U}$, then we get

$$
(\bar{\Omega}_V, \bar{\omega}_V)|_{U\cap V} = s_V^* i^*(\Omega, \omega)|_{U\cap V} \sim s_V^* \pi^*(\bar{\Omega}_U, \bar{\omega}_U)|_{U\cap V} = (\bar{\Omega}_U, \bar{\omega}_U)|_{U\cap V}
$$

By the construction described in the first paragraph, we obtain a unique l.c.s. structure $(\bar{\Omega}, \bar{\omega})$ on $P_L$, such that $(\bar{\Omega}, \bar{\omega})|_U \sim (\bar{\Omega}_U, \bar{\omega}_U)$ for all $U \in \mathcal{U}$. Moreover, we have $\pi^*(\bar{\Omega}, \bar{\omega})|_{\pi^{-1}(U)} \sim i^*(\Omega, \omega)|_{\pi^{-1}(U)}$ for all $U \in \mathcal{U}$ and hence $\pi^*(\bar{\Omega}, \bar{\omega}) \sim i^*(\Omega, \omega)$. Uniqueness is also clear for if $(\Omega', \omega')$ is any l.c.s. structure on $P_L$ with $\pi^*(\Omega', \omega') \sim i^*(\Omega, \omega)$, then

$$
(\Omega', \omega')|_U = s_U^* \pi^*(\Omega', \omega') \sim s_U^* i^*(\Omega, \omega) = (\bar{\Omega}_U, \bar{\omega}_U),
$$

and so $(\Omega', \omega') \sim (\bar{\Omega}, \bar{\omega})$. □

Suppose $(M, \Omega, \omega) \sim (M, \Omega', \omega')$ with $\Omega' = (1/a)\Omega$ and $\omega' = \omega + (\mathrm{d}a/a)$. Then $\mathrm{Diff}^\infty(M, \Omega, \omega) = \mathrm{Diff}^\infty(M, \Omega', \omega')$ and $\mathrm{Ham}(M, \Omega, \omega) = \mathrm{Ham}(M, \Omega', \omega')$. Moreover, $\psi$ is an equivariant moment mapping for $(\Omega, \omega)$ iff $(1/a)\psi$ is an equivariant moment

mapping for $(\Omega', \omega')$. Since the orthogonal relation is the same for both l.c.s. structures a submanifold $L$ satisfies the conditions of Theorem 1 for $(\Omega, \omega)$ iff it does for $(\Omega', \omega')$. Moreover, we have $\mathfrak{g}'_L = \mathfrak{g}_L$. So the reduced spaces $(P_L, \Omega, \omega)$ and $(P'_L, \Omega', \omega')$ are the same, up to conformal equivalence. Let $a \in C^\infty(M, \mathbb{R})$ be such that $\omega = (\mathrm{d}a/a)$, locally around $x_0 \in L$. Then $L$ locally given as $\{x : (1/a)\psi(x) = (1/a)\psi(x_0)\}$. If $(M, \Omega, \omega)$ is symplectic, i.e. $\omega = 0$, then this construction yields of course ordinary symplectic reduction [3,7].

## 6. Examples

Let $G$ be a discrete group acting freely and properly on an l.c.s. manifold $(M, \Omega, \omega)$, such that $l_g \in \mathrm{Diff}^\infty(M, \Omega, \omega)$ for all $g \in G$. Then $\mathfrak{g} = 0$, $\psi = 0$ is an equivariant moment mapping, and the only possible choice for $L$ is $L = M$. If we choose $G_L = G$, then $P_L = M/G$ carries an l.c.s. structure. Notice that even if we start with a symplectic manifold, $M/G$ need not be conformally equivalent to a symplectic manifold. For example, let $(M, \Omega, \omega)$ be an l.c.s. manifold, let $\pi : \tilde{M} \to M$ be a normal covering such that $\tilde{\omega} := \pi^*\omega$ becomes exact, and set $\tilde{\Omega} := \pi^*\Omega$. Then $(\tilde{M}, \tilde{\Omega}, \tilde{\omega})$ is conformally equivalent to a symplectic manifold and $G$, the group of deck transformations, satisfies $l_g \in \mathrm{Diff}^\infty(\tilde{M}, \tilde{\Omega}, \tilde{\omega})$ for all $g \in G$. Obviously $P_L = \tilde{M}/G = M$ and $(\bar{\Omega}, \bar{\omega}) \sim (\Omega, \omega)$. The smallest $\tilde{M}$ one can take is the covering corresponding to the kernel of the homomorphism $\pi_1(M) \to \mathbb{R}$, $\sigma \mapsto \int_\sigma \omega$.

Another example is connected with physical applications. In [9], the following l.c.s. reduction is applied in several dynamical systems. Let $(M, \Omega, \omega)$ be an l.c.s. manifold, and $f \in C^\infty(M, \mathbb{R})$ be a Hamiltonian function. One considers $F = \sharp\,\mathrm{d}f$ and a smooth level set $M^c = \{x \in M : f(x) = c\}$, and one assumes that $f$ does not vanish on $M^c$. Then, one obtains a reduced l.c.s. structure on $\hat{M}^c = M^c/F$. This structure enables to get a conformally symplectic transversal derivative cycle. Then some symmetries of the Lyapunov spectrum of this cocycle appear. For details, see [9].

## 7. Hamiltonian system on the reduced space

First, we consider the general reduction and assume the notation of Proposition 1.

**Proposition 4.** *Let $P_L$ be the reduced l.c.s. manifold by means of $(L, E)$ and let $u \in C^\infty(M, \mathbb{R})$ be a Hamiltonian such that* $\mathrm{d}^\omega u$ *vanishes on $E$* and its flow $\mathrm{Fl}_t^{\sharp\,\mathrm{d}^\omega u}$ *preserves $L$ and the bundle $E$. Then there exists a unique $\bar{u} \in C^\infty(P_L, \mathbb{R})$ such that $u$ extends $\bar{u} \circ \pi$* and $\pi_* \sharp\,\mathrm{d}^\omega u = \sharp_{P_L}\,\mathrm{d}^{\bar{\omega}}\bar{u}$. *In particular,* $\mathrm{Fl}_t^{\sharp\,\mathrm{d}^\omega u}$ *induces a flow of Hamiltonian automorphisms on $P_L$.*

**Proof.** Since $\mathrm{d}^\omega u|_E = \mathrm{d}u|_E = 0$, the $\bar{u}$ as above exists. For all $\bar{v} \in C^\infty(P_L, \mathbb{R})$ and

$v \in C^\infty(M, \mathbb{R})$ with $v|_L = \bar{v} \circ \pi$, we then get

$$(\sharp_{P_L} \mathrm{d}^{\bar{\omega}} \bar{u})(\pi(x))\bar{v} = \{\bar{v}, \bar{u}\}_{P_L}(\pi(x)) = \{v, u\}_M(x) = (\sharp \mathrm{d}^{\omega} u)(x) v$$
$$= (\sharp \mathrm{d}^{\omega} u)(x)(\bar{v} \circ \pi) = (\pi_* \sharp \mathrm{d}^{\omega} u)(x)\bar{v}.$$

Here we use that, by assumption, $\sharp \mathrm{d}^{\omega} u$ is tangent to $L$. □

Now we consider the case of Lie group actions.

**Lemma 1.** *Let $G$ and $H$ be two finite dimensional, connected Lie groups acting symplectically on an l.c.s. manifold $(M, \Omega, \omega)$ with moment mappings $\psi_G$ and $\psi_H$. Then*
1. $a_g l_g^* \psi_H = \psi_H$ *for all* $g \in G$,
2. $a_g l_g^* \hat{\psi}_H(Y) = \hat{\psi}_H(Y)$ *for all* $Y \in \mathfrak{h}$, $g \in G$,
3. $T\psi_H \cdot \zeta_X^G + \omega(\zeta_X^G)\psi_H = 0$ *for all* $X \in \mathfrak{g}$,
4. $L^{\omega}_{\zeta_X^G} \hat{\psi}_H(Y) = 0$ *for all* $Y \in \mathfrak{h}$, $X \in \mathfrak{g}$,
5. $\zeta_{\mathfrak{h}}^H(x) \subseteq \zeta_{\mathfrak{g}}^G(x)^{\perp}$ *for all* $x \in M$, *and*
6. $\{\hat{\psi}_G(X), \hat{\psi}_H(Y)\} = 0$ *for all* $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$

*are equivalent. Moreover, each statement is equivalent to the corresponding statement with the rôles of $G$ and $H$ exchanged. If these equivalent conditions are satisfied, then* $[\zeta_X^G, \zeta_Y^H] = 0$ *for all* $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$ *and the actions of $G$ and $H$ commute.*

**Proof.** From $\Omega(\zeta_X^G, \zeta_Y^H) = -L^{\omega}_{\zeta_X^G} \hat{\psi}_H(Y) = \{\hat{\psi}_G(X), \hat{\psi}_H(Y)\}$, we obtain the equivalence of (4)–(6). Since and $L_Z^{\omega} \hat{\psi}(Y) = \langle T\psi \cdot Z + \omega(Z)\psi, Y\rangle$ these are equivalent to (3). (1) and (2) are the infinitesimal versions of (3) and (4). The equivalence to the statements with the rôles of $G$ and $H$ exchanged is obvious from the symmetry of (5) or (6). From (6), we also get $[\zeta_X^G, \zeta_Y^H] = -\sharp \mathrm{d}^{\omega}\{\hat{\psi}_G(X), \hat{\psi}_H(Y)\} = 0$, i.e. the actions of $G$ and $H$ commute. □

Assume that the equivalent conditions of Lemma 1 are satisfied and suppose $L$ satisfies the assumptions of Theorem 1. Then $\zeta_Y^H$ is tangential to $L$ and thus at least $H_\circ$, the id component of $H$, leaves $L$ invariant.

**Theorem 2.** *Let $(M, \Omega, \omega)$, $G$, $L$ and $G_L$ satisfy the conditions of Theorem* 1. *Moreover, let $H$ be another finite dimensional Lie group, acting symplectically on $(M, \Omega, \omega)$ with equivariant moment mapping $\psi_H$, such that $a_g l_g^* \psi_H = \psi_H$ for all $g \in G$. Finally assume that the actions of $G$ and $H$ commute and suppose that $H$ leaves $L$ invariant. Then the $H$-action descends to a symplectic action on $P_L$, which has an equivariant moment mapping naturally induced from $\psi_H$.*

**Proof.** Since the action of $G$ and $H$ commute the action of $H$ descends to an action $\bar{l}$ on $P_L$. Moreover, we have

$$\pi^* \bar{l}_h^*(\bar{\Omega}, \bar{\omega}) = l_h^* \pi^*(\bar{\Omega}, \bar{\omega}) \sim l_h^* i^*(\Omega, \omega) = i^* l_h^*(\Omega, \omega) \sim i^*(\Omega, \omega),$$

and thus the uniqueness part of Theorem 1 yields $\bar{l}_h^*(\bar{\Omega}, \bar{\omega}) \sim (\bar{\Omega}, \bar{\omega})$, i.e. $H$ acts symplectically on $P_L$. Now let $a \in C^\infty(L, \mathbb{R}^+)$ be such that $\pi^* \bar{\Omega} = (1/a) i^* \Omega$ and $\pi^* \bar{\omega} =$

$i^*\omega + (\mathrm{d}a/a)$. From $a_g l_g^* \psi_H = \psi_H$ and $a_g l_g^* a = a$ for all $g \in G_L$, we obtain $l_g^*((1/a)\psi_H) = (1/a)\psi_H$ on $L$, and thus $(1/a)\psi_H$ descends to $\bar{\psi}_H \in C^\infty(P_L, \mathfrak{h}^*)$. Moreover, we have

$$\pi^*(\bar{a}_h \bar{l}_h^* \bar{\psi}_H) = (\pi^* \bar{a}_h) l_h^* \left(\frac{1}{a}\psi_H\right) = \mathrm{Ad}^*_{h^{-1}} \left(\frac{1}{a}\psi_H\right) = \pi^*(\mathrm{Ad}^*_{h^{-1}} \bar{\psi}_H),$$

and hence $\bar{\psi}_H$ is equivariant. Since we have

$$\pi^* \, \mathrm{d}^{\bar{\omega}} \hat{\bar{\psi}}_H(Y) = \mathrm{d}^{\pi^*\bar{\omega}} \left(\frac{1}{a}\hat{\psi}(Y)\right) = i_{\zeta_Y^H} \pi^* \bar{\Omega} = \pi^* i_{\bar{\zeta}_Y^H} \bar{\Omega},$$

$\psi_H$ is a moment mapping. □

Suppose $G$ is connected and let $L$ satisfy the conditions of Theorem 1. Let $f$ be a Hamiltonian function, $H = \mathbb{R}$ acting via the flow of $\sharp \mathrm{d}^\omega f$. Then $f$ is an equivariant moment mapping. Suppose $a_g l_g^* f = f$ for all $g \in G$, or equivalently $L^\omega_{\zeta_X^G} f = 0$ for all $X \in \mathfrak{g}$. Then by Lemma 1, all assumptions of Theorem 2 are satisfied, and one obtains an induced Hamiltonian system on $P_L$.

## References


[1] J.M. Arms, M.J. Gotay, G. Jennings, Geometric and algebraic reduction for singular momentum maps, Adv. Math. 79 (1990) 43–103.

[2] F. Guedira, A. Lichnerowicz, Géométrie des algébres de Lie locales de Kirillov, J. Math. Pure Appl. 63 (1984) 407–484.

[3] V. Guillemin, S. Sternberg, Symplectic Techniques in Physics, Cambridge University Press, Cambridge, 1984.

[4] S. Haller, T. Rybicki, On the group of diffeomorphisms preserving a locally conformal symplectic structure, Ann. Global Anal. Geom. 17 (1999) 475–502.

[5] H.C. Lee, A kind of even dimensional differential geometry and its application to exterior calculus, Am. J. Math. 65 (1943) 433–438.

[6] J.E. Marsden, T. Ratiu, Reduction of Poisson manifolds, Lett. Math. Phys. 11 (1986) 161–169.

[7] J. Marsden, A. Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Math. Phys. 5 (1974) 121–130.

[8] I. Vaisman, Locally conformal symplectic manifolds, Int. J. Math. Math. Sci. 8 (1985) 521–536.

[9] M.P. Wojtkowski, C. Liverani, Conformally symplectic dynamics and symmetry of the Lyapunov spectrum, Commun. Math. Phys. 194 (1998) 47–60.